Cauchy's root test

D.I.(H) Maths
paper-III, Infinite series
Group-B

Statement: A series $\sum u_n$ of positive terms is convergent if from and after some fixed term, $(u_n)^{1/n} < r < 1$ where r is a fixed number and the series is divergent if $(u_n)^{1/n} > 1$ and if $(u_n)^{1/n} = 1$, no definite conclusion is possible.

Proof: Case I It is given that $(u_n)^{1/n} < r$

Now raising nth power, we get

$u_n < r^n$, Now putting in it $n = 1, 2, 3, \dots$

we have, $u_1 < r$, $u_2 < r^2$, $u_3 < r^3, \dots$

Adding all we have

$$u_1 + u_2 + u_3 + \dots < r + r^2 + r^3 + r^4 + \dots$$

$$< \frac{r}{1-r}, \text{ which is independent of } n.$$
$= K$ say

$\therefore \sum u_n < K$

Hence the given series is convergent.

Case (ii) if $(u_n)^{1/n} > 1$ i.e. $u_n > 1^n = 1$

$\therefore u_n > 1$. putting $n = 1, 2, 3, \dots$, we get

$u_1 > 1$, $u_2 > 1$, $u_3 > 1, \dots$

Adding all, we have

$u_1 + u_2 + u_3 + \dots > ~~1+1+1+1+\dots~~ \; 1+1+1+1+\dots$

$> n$

Now taking limit $n \to \infty$, we see that $\sum u_n$ is divergent

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Hence the given series is convergent.

Case (II) if $(u_n)^{1/n} > 1$ i.e. $u_n > 1^n = 1$

$\therefore u_n > 1$. putting $n = 1, 2, 3, \ldots$, we get

$u_1 > 1, u_2 > 1, u_3 > 1, \ldots$

Adding all, we have

$u_1 + u_2 + u_3 + \ldots > \cancel{1+1+1+1+\ldots}\ 1+1+1+1+\ldots$

$> n$

Now taking limit $n \to \infty$, we see that

$\sum u_n$ is divergent